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UNIFIED FORMULATION FOR TRANSVERSE (BETATRON) AND LONGITUDINAL (SYNCHROTRON) OSCILLATIONS IN A SYNCHROTRON

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We develop here a formulation for the longitudinal oscillation (synchrotron oscillation or phase oscillation) in a synchrotron patterned after that for the transverse oscillation (betatron oscillation) which is easy to memorize and simple to use. To do this we first examine the features of the formulation for the transverse oscillation.

TRANSVERSE OSCILLATION

1. The physically simple variables are

$$\begin{cases} x = \text{displacement} \\ x' = \text{angle.} \end{cases} \tag{1}$$

These are not conjugate variables. Hence the (x,x')-space area $\pi\varepsilon_x$ is not conserved but proportional to p^{-1}. We shall call ε_x the emittance.

2. With this choice of the dependent variables the independent variable should, then, be the length z along the orbit so that we can write the equations as

$$\begin{cases} \dfrac{dx}{dz} = x' \\ \dfrac{dx'}{dz} = -K_x\, x \qquad K_x = K_x(z). \end{cases} \tag{2}$$

3. We can then calculate a local wavelength β_x such that the phase advance $d\mu_x$ is given by

$$d\mu_x = \frac{dz}{\beta_x} . \tag{3}$$

The oscillation wave number per turn ν_x is, then, given by

$$2\pi\nu_x = \oint d\mu_x = \oint \frac{dz}{\beta_x} . \tag{4}$$

4. For elliptical (x,x')-space area, then, the peak $\hat{x}$ and $\hat{x}'$ are given by

$$\hat{x} = \sqrt{\beta_x \varepsilon_x} , \qquad \hat{x}' = \sqrt{\frac{\varepsilon_x}{\beta_x}} . \tag{5}$$

LONGITUDINAL OSCILLATION

We consider only the adiabatic regions; the nonadiabatic region in the neighborhood of transition is omitted. The usual equations for small longitudinal oscillation can be written as

$$\begin{cases} \dfrac{d\phi}{dt} = -h\omega\Lambda \dfrac{\Delta p}{p} , \qquad \Lambda \equiv \dfrac{1}{\gamma^2} - \dfrac{1}{\gamma_t^2} , \qquad \omega \equiv \dfrac{c\beta}{R} \\ \dfrac{d\left(\frac{\Delta p}{p}\right)}{dt} = \dfrac{ev\cos\phi_s}{2\pi Rp} \psi \end{cases} \tag{6}$$

where the notation is conventional and needs no explanation.

1. The physically simple and useful variables are

$$\begin{cases} \phi = \text{phase deviation from synchronous } \phi_s \\ \dfrac{\Delta p}{p} = \text{relative momentum deviation from synchronous } p. \end{cases}$$

These are not conjugate variables. The $\left(\phi, \frac{\Delta p}{p}\right)$-space area $\pi\varepsilon_\phi$ is not conserved but proportional to p^{-1}. We shall call ε_ϕ the emittance.

2. With this choice of the dependent variables the independent variables should, then, be ζ defined by

$$d\zeta \equiv -\Lambda h\omega dt \equiv -\Lambda d\phi_{rf}. \tag{8}$$

So defined, $d\phi_{rf}$ is the rf phase advance. Now, we can write the equations as

$$\begin{cases} \dfrac{d\phi}{d\zeta} = \dfrac{\Delta p}{p} \\ \dfrac{d\left(\frac{\Delta p}{p}\right)}{d\zeta} = -\dfrac{V\cos\phi_s}{2\pi h\beta^2\gamma\Lambda}\psi \equiv -K_\phi\psi, \quad V \equiv \dfrac{ev}{mc^2}. \end{cases} \tag{9}$$

We assume the proper ϕ_s jump at transition so that $\frac{\cos\phi_s}{\Lambda}$ is always >0. Eq. (9) is similar to Eq. (2) with the exception that, now K_ϕ is a constant adiabatically independent of ζ, whereas K_x is a fast periodically varying function of z.

3. The wavelength β_ϕ is just

$$\beta_\phi = \frac{1}{\sqrt{K_\phi}} = \left(\frac{2\pi h\beta^2\gamma\Lambda}{V\cos\phi_s}\right)^{1/2}. \tag{10}$$

and the longitudinal phase advance is given by

$$d\mu_\phi = \frac{|d\zeta|}{\beta_\phi} = \frac{h|\Lambda|}{\beta_\phi}\omega dt = \left(\frac{h}{2\pi}\frac{\Lambda V\cos\phi_s}{\beta^2\gamma}\right)^{1/2}\omega dt. \tag{11}$$

The oscillation wave numer per turn ν_ϕ is $\frac{1}{2\pi}$ (phase advance for $\int\omega dt = 2\pi$), namely

$$\nu_\phi = \frac{h|\Lambda|}{\beta_\phi} = \left(\frac{h}{2\pi}\frac{\Lambda V\cos\phi_s}{\beta^2\gamma}\right)^{1/2}. \tag{12}$$

4. For elliptical $\left(\phi, \frac{\Delta p}{p}\right)$ space area, then, the peak $\hat{\phi}$ and $\widehat{\frac{\Delta p}{p}}$ are given by

$$\hat{\phi} = \sqrt{\beta_\phi \varepsilon_\phi} \;, \qquad \frac{\widehat{\Delta p}}{p} = \sqrt{\frac{\varepsilon_\phi}{\beta_\phi}} \tag{13}$$

5. The emittance ε_ϕ is given by

$$\varepsilon_\phi = \begin{cases} \hat{\phi}\,\dfrac{\widehat{\Delta p}}{p} & \text{for elliptical area} \\ \dfrac{8}{\pi^2}\,\hat{\phi}\,\dfrac{\widehat{\Delta p}}{p} & \text{for full stationary bucket} \\ \dfrac{4}{\pi}\,\hat{\phi}\,\dfrac{\widehat{\Delta p}}{p} & \text{for rectangular area} \end{cases} \tag{14}$$

UNITS

1. For transverse oscillation a set of consistent and convenient units is

$$\begin{cases} x \text{ in mm} \\ x' \text{ in mrad} \end{cases} \qquad \begin{cases} \varepsilon_x \text{ in mm-mrad} \\ \beta_x \text{ in m} \end{cases} \tag{15}$$

2. For longitudinal oscillation a set of consistent and convenient units is

$$\begin{cases} \phi \text{ in rad} \\ \dfrac{\Delta p}{p} \text{ in mil } (10^{-3}) \end{cases} \qquad \begin{cases} \varepsilon_\phi \text{ in rad-mil} \\ \beta_\phi \text{ in kilo } (10^3) \end{cases} \tag{16}$$

NUMERICAL VALUES FOR NAL BOOSTER

The measured linac beam characteristics at 200 MeV and at beam current up to 40 mA is

$$\begin{cases} \varepsilon_x = \varepsilon_y = 10 \text{ mm-mrad} \\ \dfrac{\widehat{\Delta p}}{p} = 1 \text{ mil} \end{cases} \tag{17}$$

For one turn injection into the booster the beam in the booster at injection, therefore, has

$$\begin{cases} \varepsilon_x = \varepsilon_y = 10 \text{ mm-mrad} \\ \varepsilon_\phi = 4 \text{ rad-mil (rectangular area with } \hat{\phi} = \pi) \end{cases} \tag{18}$$

Scaled by p^{-1}, at various energies we have

K.E. (GeV)	$\varepsilon_x = \varepsilon_y$ (mm-mrad)	ε_ϕ (rad-mil)
0.2	10	4
1	3.80	1.52
2	2.31	0.926
3	1.68	0.674
4	1.33	0.532
5	1.10	0.440
6	0.937	0.375
7	0.818	0.327
8	0.725	0.290

For 7 GeV operation at extraction ($\gamma = 8.46$ $\beta = 0.993$) we have

$$\Lambda = -0.0197 \qquad \gamma_t = 5.446$$

$$\cos\phi_s = -1 \qquad \phi_s = \pi$$

$$V = 1.066 \times 10^{-4} \qquad \underline{\text{if } v = 100 \text{ kV}}$$

and

$$\beta_\phi = 0.902 \text{ kilo.} \tag{19}$$

Using $\varepsilon_\phi = 0.327$ rad-mil at 7 GeV we have

$$\begin{cases} \hat{\phi} = 0.543 \text{ rad} & \dfrac{\widehat{\Delta p}}{p} = 0.602 \text{ mil} \\ \nu_\phi = 1.83 \times 10^{-3} \end{cases}$$

Remembering that $\beta_\phi \propto v^{-1/2}$ we see that

$$\hat{\phi} \propto v^{-1/2} \qquad \frac{\widehat{\Delta p}}{p} \propto v^{1/2} \qquad \nu_\phi \propto v^{1/2} \tag{20}$$

When we have a definite final operating energy and a definite rf voltage program [$v = v(t)$] a table giving β_ϕ at various times (or energy) during the acceleration cycle should be computed.